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A CHARACTERIZATION OF PGL$(2, q)$, q ODD

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ABSTRACT

Let G be a finite group and A and B solvable subgroups of G, such that $G = AB$ and 2 is the only common prime divisor of A and B. Under suitable restrictions of the 2-structure of A and B, it is shown that either G is solvable or G contains a solvable normal subgroup N, such that G/N contains a normal subgroup, which is isomorphic to PGL$(2, q)$, q odd.

Let G be a finite group and A and B solvable subgroups of G satisfying $G = AB$. It is an open problem to describe the structure of G.

By a famous theorem of Wielandt and Kegel, G is solvable, if A and B are nilpotent. There are various other theorems describing similar conditions, which force G to be solvable. However, there are non-solvable groups, which are the product of two solvable subgroups. Using the classification of the finite simple groups, Arad's student Fisman recently determined those finite simple groups, which are a product of two solvable subgroups of coprime orders, see [4]. If A and B have coprime orders, then we have $N = (N \cap A)(N \cap B)$ for any normal subgroup N of G; this allows one to determine all composition factors of a finite group, which is a product of two solvable subgroups of coprime orders.

If there is a prime number dividing both $|A|$ and $|B|$, then there might exist normal subgroups N of G, such that N is not the product of its subgroups $N \cap A$ and $N \cap B$. Indeed, if you take $G = \mathrm{PGL}(2, p^e)$, p odd, A the normalizer of a Sylow p-subgroup of G and B a cyclic subgroup of order $p^e + 1$, then you get $G = AB$. On the other hand, this factorization does not intersect the normal subgroup $N = \mathrm{PSL}(2, p^e)$ in a factorization of N. This is just the example which we want to characterize. More generally, we want to prove:

THEOREM. *Let G be a finite group, which contains subgroups A and B, satisfying the following conditions*:

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(a) $G = AB$.
(b) $(|A|, |B|) = 2^k$ *for some* k.
(c) *The group* B *is a nilpotent group and has a cyclic Sylow* 2*-subgroup.*
(d) *The factor group* $A/O_2(A)O(A)$ *is a cyclic* 2*-group.*

If $S(G)$ *denotes the largest solvable normal subgroup of* G, *then either* $G = S(G)$ *or there exists an odd prime power* q, $q > 3$, *such that* $\mathrm{PGL}(2,q) \leqq G/S(G) \leqq \mathrm{P\Gamma L}(2,q)$ *holds.*

This is a generalization of a theorem of Finkel and Ward [3], who considered finite groups G, satisfying the hypothesis of the theorem above and in addition $[A : O_2(A)O(A)] \leqq 2$. They proved that such a group is either solvable or $G/S(G)$ contains a normal subgroup, which is isomorphic to $\mathrm{PGL}(2,q)$ with $q \equiv 3 \pmod 4$.

As in [3], we can apply this theorem to products of supersolvable and nilpotent groups and prove:

COROLLARY. *Let* $G = AB$ *be a finite group with* A *a supersolvable and* B *a nilpotent subgroup. Assume that* $(|A|, |B|)$ *is a power of* 2, *and that* $O_2(B)$ *and the Sylow* 2*-subgroup of* A/A' *are cyclic. Then either* $G = S(G)$ *or there exists an odd prime number* $p, p > 3$, *such that* $\mathrm{PGL}(2,p) \leqq G/S(G) \leqq \mathrm{P\Gamma L}(2,p)$ *holds.*

1. Notation and preliminary results

Our notation is standard and can be found in [5] or [9]. Moreover, all groups considered are finite. The following results are frequently used:

LEMMA 1.1 [9; VI, 4.5]. *If* $G = AB$, $A \cap B = 1$ *and* $x \neq 1$ *is an element of* A, *then* x *is not conjugate to an element of* B.

LEMMA 1.2. *If* $G = AB$ *with* A *and* B *cyclic* 2*-groups, then the commutator subgroup* G' *of* G *is an abelian group of rank at most* 2. *The sectional rank of* G *is at most* 4.

PROOF. By a theorem of Ito [9; VI, 4.4], the commutator subgroup G' of G is an abelian group, generated by at most two elements, see [8; Satz 17]. Since the rank of the abelian group G/G' is clearly at most 2, the sectional rank of G evidently is at most 4.

LEMMA 1.3. *Let* G *be a* 2*-group. Suppose that there exists an involution* j *in* G *such that* $C_G(j)$ *has rank* 2. *Then* $\mathrm{SCN}_3(G) = \varnothing$; *in particular, the sectional rank of* G *is at most* 4.

PROOF. By our hypothesis, j is contained in each elementary-abelian subgroup of order 4, which is centralized by j.

Let N be an elementary-abelian normal subgroup of G and suppose $|N| \geqq 8$. Then $|C_N(j)| \geqq 4$ and hence $j \in C_N(j)$. Thus we get $N \leqq C_G(j)$, a contradiction.

We conclude that the order of an elementary-abelian, normal subgroup of G is at most 4. Now we can apply [11] and find that the sectional rank of G is at most 4.

LEMMA 1.4 [9; VI, 4.6]. *If $G = AB$ and if p is a prime number, then there exist Sylow p-subgroups S of A and T of B such that ST is a Sylow p-subgroup of G.*

PROPOSITION 1.5. *Let G be a finite group with subgroups A and B such that $G = AB$, $(|A|, |B|) = 1$, $A/O_2(A)O(A)$ is an abelian 2-group and B is a nilpotent group of odd order. Then G is solvable.*

PROOF. Using the Unbalanced Group Theorem, Finkel and Lundgren proved this in [2]. A more elementary proof can be found in [10].

PROPOSITION 1.6. *Let G be a group which satisfies the hypothesis of the theorem. Let p be an odd prime, $q = p^e > 3$ and suppose that G contains a normal subgroup M, isomorphic to* PSL(2, q), *and that G is contained in the automorphism group* PΓL(2, q) *of* PSL(2, q). *Then $O_2(A) = 1$ and G contains a normal subgroup isomorphic to* PGL(2, q).

PROOF. By inspection of the subgroups of G or by Lemma 1.5, we infer $O_2(B) \neq 1$. Clearly, $D := A \cap B$ is a 2-group. Hence $\langle (A \cap B)^G \rangle$ is contained in A. This implies $A \cap B = 1$.

Now let P be a Sylow p-subgroup of M. Since $C_G(P) = P$, a conjugate of P is contained in A; without loss P is contained in A. By inspection of the subgroups of M [9; II, 8.27], we conclude

$$N_M(P) \leqq A \leqq N_G(P)$$

and that $B \cap M$ contains a cyclic subgroup of order $(q+1)/2$. Since $|A \cap M|_2 \cdot |B \cap M|_2 < |M|_2$ holds, the index $[G : M]$ is even.

If e is odd, the result follows immediately. Consequently, we may assume e to be even. Then 2 does not divide $(q+1)/2$ and $B \cap M$ has odd order. Thus $G \setminus M$ contains an involution. Since this involution centralizes a cyclic subgroup of M of order $(q+1)/2$, this automorphism cannot be induced by a field automorphism. From the description of a Sylow 2-subgroup of PΓL(2, q) [6; lemma 2.3], it follows that this involution is induced by a diagonal automorphism; in particular, G contains a normal subgroup isomorphic to PGL(2, q).

2. Proof of the Theorem

Suppose the theorem is false, and let G be a counterexample of minimal order. If N is a non-trivial normal subgroup of G and if $\bar{G} = G/N$, then $\bar{G} = \bar{A}\bar{B}$, and $\bar{A}$ and $\bar{B}$ satisfy the hypothesis of the theorem. Because of the minimality of G, the group $\bar{G}$ satisfies the conclusion of the theorem. If X is a proper subgroup of G, which either contains A or B, then $X = (X \cap A)(X \cap B)$ and X satisfies the hypothesis and conclusion of the theorem.

By Proposition 1.5 we get $O_2(B) \neq 1$. The famous theorem of Wielandt and Kegel [9; VI, §4] implies $A \neq O_2(A)$. We begin by showing:

(2.1) The group G contains exactly one minimal normal subgroup M, which is a direct product of non-abelian simple groups. In particular $S(G) = 1$.

Proof. Let M be a minimal normal subgroup of G. Then G/M satisfies the conclusion of the theorem. Since G does not satisfy the conclusion of the theorem, M is not solvable and a direct product of non-abelian simple groups.

Assume L is a minimal normal subgroup with $L \neq M$. Since G/L as well satisfies the conclusion of the theorem, G has precisely two minimal normal subgroups, and there exist odd prime powers q and r such that $L \cong \mathrm{PSL}(2, r)$ and $M \cong \mathrm{PSL}(2, q)$ holds.

Define $P := F(M \cap A)$ the Fitting subgroup of $M \cap A$. By Proposition 1.6 P is a Sylow subgroup of M of order q. We have

$$N_G(P) = AN_B(P).$$

As L is contained in $N_G(P)$, $N_G(P)$ is non-solvable and $O_2(N_B(P)) \neq 1$ by Proposition 1.5. Let b be the involution in B. Then

$$\langle b^G \rangle \leqq N_G(P)$$

and hence L is contained in $\langle b^G \rangle$.

By symmetry, M as well is contained in $\langle b^G \rangle$. But now

$$M \leqq \langle b^G \rangle \leqq N_G(P).$$

This contradiction completes the proof of (2.1).

(2.2) We have $AM = G = BM$. In particular, G/M is a 2-group.

Proof. Assume $AM < G$. Then $AM = A(B \cap AM)$. By the minimality of G, there exists an odd prime power q such that $M \cong \mathrm{PSL}(2, q)$ and AM contains a normal subgroup isomorphic to $\mathrm{PGL}(2, q)$. Thus

$$G \leqq \mathrm{Aut}(M) = \mathrm{P\Gamma L}(2, q)$$

and G is no counterexample.

A similar argument shows $G = BM$. As $(|A|, |B|)$ is a power of 2, the factor group G/M is a 2-group.

(2.3) Suppose $A \leqq L < G$. Then $L \cap B$ has odd order and L is solvable. In particular we have $A \cap B = 1$.

PROOF. Assume $L \cap O_2(B) \neq 1$. The normal closure of $L \cap O_2(B)$ is contained in L. Since G contains exactly one minimal normal subgroup, M is contained in L, contradicting (2.2).

We proved $L \cap O_2(B) = 1$. By Proposition 1.5, L is solvable.

(2.4) We have $O_2(A) = 1$.

PROOF. Assume $O_2(A) \neq 1$. By Lemma 1.4, there exists a Sylow 2-subgroup S of A such that $SO_2(B)$ is a Sylow 2-subgroup of G. Let b be an involution in B. Then b normalizes S, and if X is a $\langle b \rangle$-invariant normal subgroup of S contained in $O_2(A)$, then $X = 1$ by (2.3). In particular, S is abelian and $O_2(A) \cap O_2(A)^b = 1$. Hence $O_2(A)$ is cyclic and S has rank 2.

Let j be the involution in $O_2(A)$. Again by (2.3), S is a Sylow 2-subgroup of $C_G(j)$ and $C_G(j)$ is solvable; more exactly we get $C_G(j) = SO(C_G(j))$.

We now claim that the sectional rank of a Sylow 2-subgroup of G is at most 4 and that the minimal normal subgroup M is simple. If $O_2(B)$ has order 2, then the sectional rank of a Sylow 2-subgroup of G is at most 3, and the assertion is trivial. Hence we may assume $|O_2(B)| > 2$ and choose $x \in B$ an element of order 4. Since $j^G \cap S = \{j, j^b\}$ and x normalizes $\langle S, b \rangle$, we get $j^x \in \langle S, b \rangle \setminus S$ and $|O_2(A)| = 2$. Furthermore $SS^x = \langle S, b \rangle$ and

$$Z := S \cap S^x = Z(\langle S, b \rangle)$$

is a cyclic group such that $S = \langle j \rangle \times Z$ and $ZO_2(B)$ is a Sylow 2-subgroup of $C_G(b)$. We apply Lemma 1.3 to the action of j on $ZO_2(B)$ and conclude that the sectional rank of a Sylow 2-subgroup of G is at most 4. By Lemma 1.1, the involution j is not conjugate to an involution of $ZO_2(B)$; an application of Thompson's transfer lemma yields $j \notin O^2(G)$. By the solvability of $C_G(j)$, each direct factor of M is normalized by j. Now the precise structure of $C_G(j)$ implies the simplicity of M.

In any case, we have seen that M is a simple group, and the sectional rank of a Sylow 2-subgroup is at most 4. By [7], M is a known group. Since j is an involution with a solvable centralizer and $O(C_G(j)) \neq 1$, M can only be one of the following groups: PSL$(2, q)$, PSL$(3,3)$, PSU$(3,3^2)$, PSp$(4,3)$, $G_2(3)$, PSL$(3,4)$, A_7, A_9 or A_{11}. Using the precise structure of $C_G(j)$, one can exclude each of these groups; details are omitted and can be found in [2; lemma 3.2]. This completes the proof of (2.4).

(2.5) M is a simple group.

PROOF. Since $O_2(A) = 1$, a Sylow 2-subgroup S of G is a product of two cyclic subgroups. Hence S' is abelian and the sectional rank of S is at most 4. In particular, the number of direct factors of M is at most 2.

Assume $M = M_1 \times M_2$ is the direct product of two factors. Since S interchanges M_1 and M_2 and since S' is abelian, the Sylow 2-subgroups of M_1 and M_2 are elementary-abelian of order 4; hence $M_1 \cong$ PSL$(2, p^e)$ with $p^e \equiv 3, 5 \pmod 8$. A Sylow p-subgroup of M is self-centralizing in G, hence p divides $|A|$. Thus $O_p(A)$ is a Sylow p-subgroup of M and $N_G(O_p(A)) = A$. Since A has cyclic Sylow 2-subgroups, we infer $p^e \equiv 3 \pmod 4$, and the order of a Sylow 2-subgroup of A does not exceed 4. But then S is a product of a cyclic subgroup and a cyclic subgroup of order 4, forcing the sectional rank of S to be at most 3. This contradiction shows the simplicity of M.

(2.6) M is a Chevalley group of odd characteristic and belongs to the list in [7].

PROOF. By Lemma 1.2, a Sylow 2-subgroup of M has sectional rank at most 4, and its commutator subgroup is an abelian group of rank at most 2.

If the assertion is false, we can extract from [7] that M is one of the following groups: PSL$(2,8)$, PSL$(2,16)$, PSL$(3,4)$, PSU$(3,4^2)$, A_7, M_{11}, M_{12}.

Since $A \cap B = 1$, an involution of A cannot be conjugate to an involution of B; in particular, G has at least two conjugacy classes of involutions.

If M is one of the Chevalley groups, then M has only one conjugacy class of involutions; in particular either $A \cap M$ or $B \cap M$ has odd order. If 2^n is the exponent of a Sylow 2-subgroup of M, then the order of a Sylow 2-subgroup of G is bounded by $2^n \cdot [G : M]^2$. This argument excludes the Chevalley groups of characteristic 2.

In case $M \cong A_7$, we get $G = \Sigma_7$. But then a Sylow 2-subgroup of G is not a product of two cyclic subgroups.

The fact that $\mathrm{Aut}(M_{11}) \cong M_{11}$ has only one conjugacy class of involutions excludes M_{11}. In order to exclude M_{12}, one needs the information about the structure of A and B.

Thus M turns out to be a Chevalley group of odd characteristic.

(2.7) There exists an odd prime power q with $M \cong \mathrm{PSL}(2, q)$.

PROOF. By (2.6), M is a Chevalley group of odd characteristic p and belongs to the list in [7]. Since the commutator subgroup of a Sylow 2-subgroup of M is an abelian group of rank at most 2, the group M is isomorphic to one of the following groups: $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q^2)$, $G_2(q)$, ${}^3D_4(q)$ or ${}^2G_2(q)$.

As $O_2(B) \neq 1$, the prime number p divides $|A|$. By [13] $O_p(A)$ is a Sylow p-subgroup of $O(A)$. Hence A is contained in $N_G(O_p(A))$, and A contains a Sylow 2-subgroup of $N_G(O_p(A))$; in particular, the Sylow 2-subgroups of $N_M(O_p(A))$ are cyclic. This implies that M is a Chevalley group of Lie rank 1, see [1].

If $M \cong {}^2G_2(q)$, then $G = M$ and the Sylow 2-subgroups cannot be a product of two cyclic subgroups.

In order to exclude the case $M \cong \mathrm{PSU}(3, q^2)$, one can use the list of subgroups of M, see [12].

We proved $M \cong \mathrm{PSL}(2, p^e)$.

An application of Proposition 1.6 now completes the proof of the theorem.

3. Proof of the Corollary

Since a supersolvable group has a nilpotent commutator subgroup [9; VI, 9.1], the group G satisfies the hypothesis and conclusion of the theorem. Without loss, $S(G) = 1$. Then G contains a unique minimal normal subgroup, which is contained in a normal subgroup M, isomorphic to $\mathrm{PGL}(2, p^e)$ with an odd prime number p and $p^e > 3$. Now we apply Proposition 1.6 to conclude that A contains the M-normalizer of a suitable Sylow 2-subgroup of M. Thus this normalizer is supersolvable as well. This forces $e = 1$ and proves the Corollary.

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